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Uniqueness of the Cauchy Problem when the Initial Surface Contains Characteristic Points

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1. Introduction

We consider linear partial differential operators of order m in n-dimensional space,

$$P(x,D)=\sum_{|\alpha|\leq m} a^{\alpha}(x)\,D^{\alpha}, \tag{1}$$

where $x=(x_1,\ldots,x_n)$ is a point in R_n, $\alpha=(\alpha_1,\ldots,\alpha_n)$ is an n-tuple of integers $\geqq 0$ with

$$|\alpha|=\sum_{j=1}^{n}\alpha_j \quad\text{and}\quad D^{\alpha}=D_1^{\alpha_1}\ldots D_n^{\alpha_n} \quad\text{with}\quad D_j=\frac{\partial}{\partial x_j}.$$

The principal part $P_m(x,D)$ is the homogeneous part of order m,

$$P_m(x,D)=\sum_{|\alpha|=m} a^{\alpha}(x)\,D^{\alpha}. \tag{2}$$

The classical Cauchy-Kovalevsky theorem asserts that locally there exists a unique analytic solution of the Cauchy problem: if a^{α} and f are analytic in a neighborhood of 0 and the coefficient of D_n^m is $\neq 0$ at 0, then for every set of functions $\varphi_j(x_1,\ldots,x_{n-1})$, $j=0,\ldots,m-1$, which are analytic in a neighborhood of 0, there exists a unique solution of the equation

$$P(x,D)\,u=f \tag{3}$$

which is analytic in a neighborhood of 0 and satisfies the conditions

$$D_n^j u=\varphi_j \qquad \text{when } x_n=0,\ j=0,\ldots,m-1. \tag{4}$$

The hyperplane $x_n=0$, on which the initial data φ_j are prescribed, is called the initial surface. The condition that the coefficient of D_n^m is $\neq 0$ at 0 means that the initial surface is not characteristic at 0.

A surface $\Phi(x)=\Phi(x^0)$, where $\Phi\in C^1$ and grad $\Phi(x^0)\neq 0$ is called characteristic at x^0 with respect to the differential operator $P(x,D)$ if

$$P_m(x^0,\operatorname{grad}\Phi(x^0))=0. \tag{5}$$

If, in addition, for some j,

$$P_m^{(j)}(x^0,\operatorname{grad}\Phi(x^0))\neq 0, \tag{6}$$

where $P_m^{(j)}(x,\xi)=(\partial/\partial\xi_j)\,P_m(x,\xi)$, then the surface is called simply characteristic at x^0.

The Cauchy-Kovalevsky theorem is also valid when the initial surface is an analytic surface passing through an arbitrary point x^0 at which it is assumed not characteristic with respect to $P(x,D)$. In this paper we are concerned with the uniqueness assertion of this theorem.

In 1901, HOLMGREN [*1*] proved that there can be at most one solution of the Cauchy problem even in the class of C^m functions. HÖRMANDER [*2*, Theorem 5.3.1] proved HOLMGREN's uniqueness theorem even when distribution solutions are allowed: Let $P(x,D)$ have analytic coefficients in an open neighborhood U of a point $x^0 \in R_n$, let Φ be a real-valued function in $C^1(U)$ and suppose that

$$P_m(x^0, \operatorname{grad} \Phi(x^0)) \neq 0. \tag{7}$$

Then there exists a neighborhood $U' \subset U$ of x^0 such that every distribution u in U satisfying the equation $P(x,D)u=0$ and vanishing when $\Phi(x) > \Phi(x^0)$, $x \in U$, must also vanish in U'.

The theorem asserts that if u is a distribution solution of $P(x,D)u=0$ defined in a neighborhood of x^0 and vanishing on one side of the surface $\Phi(x)=\Phi(x^0)$, it must also vanish in a whole neighborhood of x^0. Note that U' depends on U and not on the particular distribution u. Condition (7) requiring that the surface is not characteristic at x^0 cannot be dropped entirely. In fact if an analytic surface $\Phi(x)=\Phi(x^0)$ is simply characteristic at every one of its points in a neighborhood of x^0, then there is a C^m solution of $P(x,D)u=0$ defined in a neighborhood of x^0, vanishing on one side of the surface and such that x^0 belongs to the support of u. This theorem, originally due to GOURSAT, was shown by HÖRMANDER [*2*] to be valid for every characteristic plane, not necessarily simply characteristic, when the differential operator has constant coefficients.

The next question is this. Suppose that condition (7) is not satisfied, that is, the initial surface is (simply) characteristic at x^0, but there is no neighborhood of x^0 such that the piece of the surface contained in it is characteristic at every one of its points. What are the conditions under which the conclusion of HOLMGREN's theorem is still valid, that is, uniqueness of the Cauchy problem still holds? The first answer to this question was given by HÖRMANDER [*2*] who showed that condition (7) can be replaced by a convexity condition. For differential operators with constant coefficients, more general conditions were obtained by TRÈVES [*3*]. The main idea in his proof is the same as the one used by HÖRMANDER.

In this paper we consider only differential operators with constant coefficients and present two theorems. The first theorem contains and extends the result of TRÈVES, and the second yields as a corollary a new local uniqueness result for the wave equation. While the basic idea in the proof is also the same as that used originally by HÖRMANDER, the proof is simpler than the one given by TRÈVES.

Before closing this introduction, we describe an important application of theorems concerning the uniqueness of the Cauchy problem. The differential equation $P(D)u=f$ may be locally solvable at each point of a domain without being globally solvable in the domain. It was shown by MALGRANGE [*4*] that the equation has a distribution solution u in a domain Ω for every $f \in C^\infty(\Omega)$ if and only if Ω is P-convex. An open set Ω is P-convex if to every compact set $K \subset \Omega$ there is another compact set $K' \subset \Omega$ such that if u is any distribution with compact support in Ω and $\operatorname{supp} P(-D)u \subset K$ then $\operatorname{supp} u \subset K'$. Note that K' is a fixed compact set depending on K and not on the particular distribution u. If u is a distribution with compact support in R_n, the convex hull of supp $P(-D)u$ coincides with the convex hull of supp u [2, Lemma 3.4.3]. It follows that every open convex set is P-convex. It also follows that an open set Ω is P-convex if uniqueness of the Cauchy problem for $P(-D)$ holds in a neighborhood of every point of its boundary, *i.e.*, if at every point x^0 of the boundary of Ω the following assertion is valid: to each neighborhood U of x^0 there exists a neighborhood $U' \subset U$ of x^0 such that every distribution u in U satisfying the equation $P(-D)u=0$ and vanishing outside of Ω must also vanish in U'.

As an example, it follows from HOLMGREN's uniqueness theorem that if Ω is an open set the boundary of which is a C^1 surface containing no points at which it is characteristic with respect to $P(-D)$, then Ω is P-convex. Similarly, more general sufficient conditions for P-convexity follow from the uniqueness results of this paper. Thus, Ω is P-convex if its boundary is a sufficiently smooth surface satisfying the conditions described in Section 2 at every point where it is characteristic with respect to $P(-D)$. The precise statements are easy to formulate, and they will not be presented here.

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2. Statement and Discussion of Results

Theorem 1. *Let $P(D)$ be a differential operator having real coefficients in the principal part. Let U be a neighborhood of a point $x^0 \in R_n$ and Φ a real-valued function in $C^k(U)$, where k is an integer $\geqq 2$, such that* grad $\Phi(x^0)=N \neq 0$ *and*

$$P_m(N)=0, \quad \big(P_m^{(1)}(N), \dots, P_m^{(n)}(N)\big)=L \neq 0, \tag{8}$$

$$D_L^i[\Phi(x)-\Phi(x^0)]_{x=x^0} \begin{cases} =0 & \text{for } i=0, \dots, k-1, \\ \neq 0 & \text{for } i=k, \end{cases} \tag{9}$$

$$D_L^k[\Phi(x)-\Phi(x^0)]_{x=x^0}>0 \qquad \text{when } k \text{ is even}, \tag{10}$$

where D_L denotes the directional differentiation operator in the direction of L. Suppose furthermore that the following condition is satisfied:

(A) *There are integers $l \geqq 1$ and $p \geqq 2$ such that*

$$D_L^i[\operatorname{grad} \Phi(x)-\operatorname{grad} \Phi(x^0)]_{x=x^0}=0 \qquad \text{for } i=0, \dots, l-1,$$

$$P_m(\tau \xi+N)=0(\tau^p) \qquad \text{as } \tau \to 0, \ \tau \geqq 0,$$

for all unit vectors ξ normal to N and L, and such that either (a) *k is even and $pl \geqq k$ or* (b) *k is odd and $pl \geqq k+1$. Then there exists a neighborhood $U' \subset U$ of x^0 such that every distribution u in U satisfying the equation $P(D)u=0$ and vanishing when $\Phi(x)>\Phi(x^0)$, $x \in U$, must also vanish in U'.*

Condition (8) means that the surface $\Phi(x)=\Phi(x^0)$ is simply characteristic at x^0. Note that the vectors N and L are normal. This follows from Euler's identity for homogeneous polynomials,

$$\sum_{j=1}^{n} N_j P_m^{(j)}(N)=m P_m(N)=0.$$

Condition (9) means that the bicharacteristic line through (x^0, N) (the line through x^0 and parallel to L) is tangent of order $k-1$ to the surface $\Phi(x)=\Phi(x^0)$ at x^0. Condition (10) means that, when k is even, in a sufficiently small deleted neighborhood of x^0 the bicharacteristic lies in the set where $\Phi(x)>\Phi(x^0)$, that is, on the side of the surface where u is assumed to vanish.

Concerning condition (A) we note that $l=1$ implies no additional restriction on Φ, and $p=2$ implies no additional restriction on P_m. In fact,

$$P_m(\tau\xi+N)=P_m(N)+\tau\sum_{j=1}^{n}\xi_j P_m^{(j)}(N)+(\tfrac{1}{2})\tau^2\sum_{j,k=1}^{n}\xi_j\xi_k P_m^{(j,k)}(N)+\cdots,$$

and since ξ is assumed normal to L,

$$P_m(\tau\xi+N)=(\tfrac{1}{2})\tau^2\sum_{j,k=1}^{n}\xi_j\xi_k P_m^{(j,k)}(N)+\cdots.$$

From these observations the following corollaries follow immediately.

Corollary 1. *When $k=2$, Theorem 1 is valid with condition* (A) *omitted.*

This result is also a special case of Theorem 5.3.2 of HÖRMANDER [2] concerning differential operators with variable coefficients.

Corollary 2. *Theorem 1 is valid with condition* (A) *replaced by the following condition*:

(A_1) $D_L^i[\operatorname{grad}\Phi(x)-\operatorname{grad}\Phi(x^0)]_{x=x^0}=0$ *for* $i=0,\ldots,[(k-1)/2]$.

This result was first obtained by TRÈVES [3, Theorem 6.9].

Corollary 3. *Theorem 1 is valid with condition* (A) *replaced by the following condition*:

(A_2) *There is an even integer $p\geqq k$ such that*

$$P_m(\tau\xi+N)=O(\tau^p), \qquad \text{as } \tau\to 0,\ \tau\geqq 0,$$

for all unit vectors ξ normal to N and L.

Note that condition (A) imposes restrictions on Φ and on P_m which are complementary and that condition (A_1) is condition (A) with $p=2$, while condition (A_2) is condition (A) with $l=1$.

Example. Let $P(D)$ be a differential operator in R_4 with principal part

$$P_4(D)=-D_3D_4^3+D_1^4-D_2^4.$$

Let $x^0=0$, and

$$\Phi(x)=x_3^3+x_1x_3-x_4.$$

Here $\operatorname{grad}\Phi(0)=N=(0,0,0,-1)$, and $L=(0,0,1,0)$. Thus the surface $\Phi(x)=0$ is simply characteristic at 0 with respect to $P(D)$, the bicharacteristic through $(0,N)$ coincides with the x_3 axis and $D_L=D_3$. In this case $k=3$ and $p=4$, so that condition (A_2) is satisfied and uniqueness of the Cauchy problem holds. However, condition (A_1) is not satisfied.

It should be noted that condition (10) cannot be dropped. In fact, when $k=2$, it was shown by MALGRANGE [5] that if conditions (8) and (9) hold and (10) is replaced by

$$D_L^k[\Phi(x)-\Phi(x^0)]_{x=x^0}<0, \tag{11}$$

then there is a C^m solution of $P(D)u=0$ defined in a neighborhood of x^0, which vanishes when $\Phi(x)>\Phi(x^0)$ and such that x^0 belongs to the support of u. The proof is based on the fact that condition (11) means that in a deleted neighborhood of x^0 the bicharacteristic through (x^0, N) lies on the set where $\Phi(x)<\Phi(x^0)$. Consequently, it can be shown that there is a surface tangent to $\Phi(x)=\Phi(x^0)$ at x^0 which is simply characteristic at each of its points and situated on the set where $\Phi(x)\leqq\Phi(x^0)$. The conclusion follows from the non-uniqueness result mentioned in the introduction. For the case when k is even and $\geqq 2$ TRÈVES [3] proved the same result, assuming the additional condition (A_1).

Theorem 2. *Theorem 1 is valid with condition* (A) *replaced by the following condition*:

(B) *For all unit vectors ξ normal to N and L and all sufficiently small $\tau\geqq 0$, either* (a) *k is even and $P_m(\tau\xi+N)$ does not change sign, or* (b) *k is odd and $P_m(\tau\xi+N)$ has the sign of $D_L^k[\Phi(x)-\Phi(x^0)]_{x=x^0}$ wherever it is not zero.*

An immediate consequence is the following uniqueness result for second order hyperbolic differential operators with constant coefficients.

Corollary 4. *Let $P(D)$ be a differential operator having as principal part the wave operator,*

$$P_2(D)=D_1^2+\cdots+D_{n-1}^2-D_n^2.$$

Let U be a neighborhood of a point $x^0\in R_n$ and Φ a real-valued function in $C^k(U)$, where k is an integer $\geqq 2$, such that grad $\Phi(x^0)=N\neq 0$, *the surface $\Phi(x)=\Phi(x^0)$ is simply characteristic at x^0, and*

$$D_L^i[\Phi(x)-\Phi(x^0)]_{x=x^0}\begin{cases}=0 & \text{for } i=0,\dots,k-1,\\ >0 & \text{for } i=k,\end{cases} \tag{12}$$

where $L=(P_m^{(1)}(N),\dots,P_m^{(n)}(N))$ and D_L denotes the directional differentiation operator in the direction of L. Then there exists a neighborhood $U'\subset U$ of x^0 such that every distribution u in U satisfying the equation $P(D)u=0$ and vanishing when $\Phi(x)>\Phi(x^0)$, $x\in U$, must also vanish in U'.

For the proof we note that if $N=(N_1,\dots,N_n)\neq 0$ and $P_m(N)=N_1^2+\cdots+N_{n-1}^2-N_n^2=0$, then $N_n^2\neq 0$. Furthermore $L=2(N_1,\dots,N_{n-1},-N_n)$, and if $\xi=(\xi_1,\dots,\xi_n)$ is normal to N and L, it must also be normal to $2N-L=(0,\dots,0,4N_n)$ so that $\xi_n=0$. Now

$$P_2(\tau\xi+N)=\tau^2(\xi_1^2+\cdots+\xi_{n-1}^2-\xi_n^2)=\tau^2(\xi_1^2+\cdots+\xi_{n-1}^2)\geqq 0.$$

Thus condition (B) of Theorem 2 is satisfied.

Condition (12) means that the bichacracteristic line through (x^0, N) (the line through x^0 and parallel to L) is tangent of finite order $k-1$ to the surface $\Phi(x)=\Phi(x^0)$ at x^0 and that, in a sufficiently small deleted neighborhood of x^0, when k is even the bicharacteristic lies on the side of the surface where u is assumed to vanish, and when k is odd the "forward" bicharacteristic half-line (the ray with vertex at x^0 in the direction of L) lies on the side of the surface where u is assumed to vanish.

Before closing this section we remark that the function Φ may be replaced by any other C^k function Φ' provided Φ and Φ' define the same surface passing through x^0 and the same "side". In fact, if Φ and Φ' are two C^k functions with non-vanishing gradient in U and if the sets $\{x: x\in U, \Phi(x)<\Phi(x^0)\}$ and $\{x: x\in U, \Phi'(x)<\Phi'(x^0)\}$ are identical, then

$$\Phi'(x)-\Phi'(x^0)=g(x)[\Phi(x)-\Phi(x^0)]$$

where $g\in C^k$ in $\{x: x\in U, \Phi(x)\neq\Phi(x^0)\}$, $g\in C^{k-1}$ in U and $g>0$. Furthermore, if D^k denotes any differentiation of order k, the function

$$[D^k g(x)][\Phi(x)-\Phi(x^0)],$$

defined when $\Phi(x)\neq\Phi(x^0)$, can be defined as a continuous function in U vanishing when $\Phi(x)=\Phi(x^0)$. Thus if Φ satisfies any of the conditions stated in this section, Φ' also satisfies the same conditions.

3. Proof of Theorem 1

The proof is based on the following lemma due to HÖRMANDER [2, Lemma 3.5.2].

Lemma. *In an open set $\Omega\subset R_n$ let $P(x,D)$ have analytic coefficients, and assume that the coefficient of D_n^m never vanishes in Ω. If u is a distribution in Ω satisfying the equation $P(x,D)u=0$ in $\Omega_c=\{x: x\in\Omega, x_n<c\}$* for some c *and if $\Omega_c\cap\operatorname{supp} u$ is relatively compact in Ω, then $u=0$ in Ω_c.*

We first give an outline of the proof of the theorem. Clearly we can choose the coordinates in such a way that $x^0=0$ and N is in the direction of $(0, \dots, 0, -1)$. Using the hypothesis and the fact that we can choose U as small as we need, we will construct in a neighborhood of $\overline{U}$ a real-valued analytic function of the form

$$F(x)=f(x')-x_n, \quad f(0)=0, \quad x'=(x_1, \dots, x_{n-1}),$$

satisfying the following conditions:

(13) There is a number $c>0$ such that the set

$$K=\{x: x\in U, \Phi(x)\leqq\Phi(x^0), F(x)\geqq -c\}$$

is a compact subset of U.

(14) $$P_m(\operatorname{grad} F(x))\neq 0 \quad \text{for all } x\in U.$$

We then make the analytic change of variables

$$y_j=x_j, \quad j=1, \dots, n-1; \quad y_n=-F(x).$$

The inverse substitution is also analytic. Let u', Ω, P', and K' be the images of u, U, P and K respectively. Condition (14) means that the level surfaces $F(x)=$ const are not characteristic with respect to P in U. Hence their images, which are the hyperplanes $y_n=$const, are not characteristic with respect to P' in Ω. Furthermore, since K is a compact subset of U, its image $K'=\{y: y\in\Omega, y_n\leqq c\}\cap\operatorname{supp} u'$ is a compact subset of Ω. It follows from the Lemma that $u'=0$ in $\{y$:

$y \in \Omega$, $y_n < c\}$, and therefore $u=0$ in

$$U' = \{x \colon x \in U,\ F(x) > -c\}$$

which is a neighborhood of 0 since $F(0)=0>-c$. In order to complete the proof, it remains only to construct a function F possessing the required properties.

Since N is in the direction of $(0, \ldots, 0, -1)$ and L is normal to N, it follows that $P_m^{(n)}(N)=0$. Therefore we can make a linear change of variables involving only $x_1, \ldots, x_{n-1}$ such that L is in the direction of $(0, \ldots, 0, 1, 0)$. Thus the bicharacteristic through $(0, N)$ coincides with the x_{n-1} axis and $D_L = D_{n-1}$. For convenience we introduce the notation $x'' = (x_1, \ldots, x_{n-2})$ and $t = x_{n-1}$ so that x is identified with (x'', t, x_n).

First we obtain an estimate for the function Φ. If U is a sufficiently small neighborhood of 0, we may solve the equation

$$\Phi(x'', t, x_n) = \Phi(0)$$

with respect to x_n and replace it with the equation

$$x_n = \varphi(x'', t)$$

where $\varphi(0,0)=0$ and $\varphi \in C^k(U)$. If $\Phi'(x'', t, x_n) = \varphi(x'', t) - x_n$, then the sets $\{x \colon x \in U,\ \Phi(x) < \Phi(0)\}$ and $\{x \colon x \in U,\ \Phi'(x) < 0\}$ are identical, and in view of the remark at the end of Section 2, we may replace Φ with Φ'. Thus, after dropping the prime, we may assume that Φ is of the form

$$\Phi(x'', t, x_n) = \varphi(x'', t) - x_n, \qquad \varphi(0,0) = 0. \tag{15}$$

Furthermore, since grad $\Phi(0)$ is in the direction of $(0, \ldots, 0, -1)$, we also have

$$\varphi_{x_j}(0,0) = 0, \quad j = 1, \ldots, n-2; \qquad \varphi_t(0,0) = 0. \tag{16}$$

Now, we expand $\varphi(x'', t)$ in a finite Taylor series,

$$\varphi(x'', t) = Q_0(t) + \sum_{j=1}^{n-2} Q_j(t)\, x_j + Q(x'', t) + o\big(|x''|^2 + |t|^k\big),$$

where $Q_0(t)$ is a polynomial of degree $\leqq k$ in t, the $Q_j(t)$, $j=1, \ldots, n-2$, are polynomials of degree $\leqq k-1$ in t and $Q(x'', t)$ is a polynomial of degree $\leqq k$ in (x'', t) without terms of degree $\leqq 1$ with respect to x''. It follows from (16) that

$$Q_j(0) = 0, \qquad j = 1, \ldots, n-2.$$

The assumption (9) implies that

$$Q_0(t) = \text{const}\; t^k,$$

and condition (10) requires that the constant is positive when k is even. By a contraction on t we may assume that

$$Q_0(t) = \sigma^k t^k, \qquad \sigma = \pm 1.$$

Without using condition (A) we can conclude that there are positive constants C_1, C_2 and ε_1 with $\varepsilon_1 \to 0$ as $|x''|$, $|t| \to 0$ such that

$$\varphi(x'', t) \geqq \sigma^k t^k - C_1 |t|\, |x''| - C_2 |x''|^2 - \varepsilon_1 |t|^k.$$

The restriction on Φ in condition (A) implies the estimate

$$\varphi(x'', t) \geqq \sigma^k t^k - C_1 |t|^l |x''| - C_2 |x''|^2 - \varepsilon_1 |t|^k. \tag{17}$$

In what follows we will use the estimate (17) with $l \geqq 1$ remembering that with $l=1$ the estimate holds without assuming any conditions other than (9) and (10).

We take as neighborhood of the origin the cylindrical set,

$$U = \{x\colon (x'', t) \in U_0,\ -z_0 < x_n < z_0\}$$

where

$$U_0 = \{(x'', t)\colon |x''| < x_0,\ -t_1 < t < t_0\},$$

and set

$$F(x) = f(x'', t) - x_n$$

with

$$f(x'', t) = \sigma\,\delta\,t - \frac{|x''|^2}{\varepsilon}.$$

The positive constants x_0, t_0, t_1, z_0, δ and ε will be chosen sufficiently small in order to satisfy conditions (13) and (14).

In order to satisfy condition (13), it is sufficient to choose the constants such that the boundaries ∂S and $\partial S'$ of the surfaces $S = \{x, x \in U, \Phi(x) = 0\}$ and $S' = \{x\colon x \in U, F(x) = -c\}$ lie on the lateral surface $\{x\colon (x'', t) \in \partial U_0,\ -z_0 < x_n < z_0\}$ of the boundary of U and such that $\partial S'$ lies "below" ∂S. Since the first requirement is satisfied simply by choosing z_0 large in comparison to x_0, t_0 and t_1, we concentrate on the second, more difficult one. We must choose the constants such that on $\partial S'$ the condition $\Phi(x) > 0$ holds.

On $\partial S'$ we have

$$\Phi(x) = \Phi(x) - [F(x) + c] = \varphi(x'', t) - \sigma\,\delta\,t + \frac{|x''|^2}{\varepsilon} - c,$$

and by use of the estimate (17)

$$\Phi(x) \geqq \sigma^k t^k - C_1 |t|^l |x''| - C_2 |x''|^2 - \varepsilon_1 |t|^k - \sigma\,\delta\,t + \frac{|x''|^2}{\varepsilon} - c.$$

Moreover, if ε is sufficiently small, we have the estimate

$$\Phi(x) \geqq \sigma^k t^k - C_1 |t|^l |x''| - \varepsilon_1 |t|^k - \sigma\,\delta\,t + \frac{|x''|^2}{(2\varepsilon)} - c.$$

First, we consider k even and set

$$\delta = t_0^{k-1+(\frac{1}{4})}, \qquad x_0 = t_0^{k-l+1/(2p)}, \qquad \varepsilon = t_0^{k-2l+(1/p)}, \qquad t_1 = t_0,$$

where l is restricted to integers $\leqq k/2$ so that $\varepsilon \to 0$ as $t_0 \to 0$. (This restriction on l is compatible with condition (A).) We state once and for all that all the following estimates are valid provided t_0 is sufficiently small. On the parts of $\partial S'$ on which $|x''| \leqq x_0$, $t = \pm t_0$ we have

$$\begin{aligned}\Phi(x) &\geqq t_0^k - C_1 t_0^l x_0 - \varepsilon_1 t_0^k - \delta t_0 - c \\ &= t_0^k [1 - C_1 t_0^{1/(2p)} - \varepsilon_1 - t_0^{\frac{1}{4}}] - c > (\tfrac{1}{2})\, t_0^k - c.\end{aligned}$$

On the parts of $\partial S'$ on which $|x''|=x_0$, $-t_1 \leqq t \leqq t_0$, we have

$$\begin{aligned}\Phi(x) &\geqq (1-\varepsilon_1)\, t^k - C_1 |t|^l x_0 - \delta |t| + \frac{x_0^2}{(2\varepsilon)} - c \\ &\geqq -C_1 t_0^{k+1/(2p)} - t_0^{k+\frac{1}{4}} + (\tfrac{1}{2})\, t_0^k - c > (\tfrac{1}{4})\, t_0^k - c .\end{aligned}$$

Thus, when k is even, condition (13) is satisfied if $c=(\frac{1}{4})\, t_0^k$ and σ is equal to either $+1$ or -1.

Next we consider k odd. We discuss only the case $\sigma=+1$. The proof for $\sigma=-1$ is obtained from the proof for $\sigma=+1$ by interchanging t_0 and t_1 throughout. We set

$$\delta = t_0^{k-1+1/(4p)}, \qquad x_0 = t_0^{k-l+1/(2p)}, \qquad \varepsilon = t_0^{k-2l+2/p+1/(4p)}, \qquad t_1 = t_0^{1+1/(4p)}$$

where l is restricted to integers $\leqq k/2+1/p$ so that $\varepsilon \to 0$ as $t_0 \to 0$. On the part of $\partial S'$ on which $|x''| \leqq x_0$, $t=t_0$ we have (as in the case of k even)

$$\Phi(x) > (\tfrac{1}{2})\, t_0^k - c .$$

On the part of $\partial S'$ on which $|x''| \leqq x_0$, $t=-t_1$ we have

$$\begin{aligned}\Phi(x) &\geqq -(1+\varepsilon_1)\, t_1^k - C_1 t_1^l x_0 + \delta t_1 - c \\ &= t_0^{k+1/(2p)} [1-(1+\varepsilon_1)\, t_0^{(k-2)/(4p)} - C_1 t_0^{l/(4p)}] \\ &> (\tfrac{1}{2})\, t_0^{k+1/(2p)} - c .\end{aligned}$$

On the part of $\partial S'$ on which $|x''|=x_0$, $-t_1 \leqq t \leqq t_0$ we have

$$\begin{aligned}\Phi(x) &\geqq -t_1^k - C_1 t_0^l x_0 - \varepsilon_1 t_0^k - \delta t_0 + \frac{x_0^2}{(2\varepsilon)} - c \\ &= (\tfrac{1}{2})\, t_0^{k-5/(4p)} - t_0^{k+k/(4p)} - C_1 t_0^{k+1/(2p)} \\ &\quad - \varepsilon_1 t_0^k - t_0^{k+1/(4p)} - c > (\tfrac{1}{4})\, t_0^{k-5/(4p)} - c .\end{aligned}$$

Thus, when k is odd, condition (13) is satified if $c=(\frac{1}{2})\, t_0^{k+1/(2p)}$.

It remains to verify condition (14). We have

$$\operatorname{grad} F = (f_{x''}, f_t, -1), \qquad f_{x''} = (f_{x_1}, \dots, f_{x_{n-2}}),$$

and for our choice of F,

$$f_{x''} = -2x''/\varepsilon, \qquad f_t = \sigma\delta .$$

Furthermore,

$$\begin{aligned}P_m(X'', T, -1) = P_m(X'', 0, -1) &+ T(\partial/\partial T)\, P_m(X'', 0, -1) + \\ &+ (\tfrac{1}{2})\, T^2 (\partial^2/\partial T^2)\, P_m(X'', 0, -1) + \cdots ,\end{aligned}$$

and since $(\partial/\partial T)\, P_m(0, 0, -1) = a$ where a is a positive constant, we have

$$P_m(X'', T, -1) = [a + R(X'', T)]\, T + P_m(X'', 0, -1)$$

where $R(X'',T)$ is a polynomial in X'', T without a constant term. Hence

$$P_m(\operatorname{grad} F) = \left[a + R\left(-\frac{2x''}{\varepsilon}, \sigma\delta\right)\right] \sigma\delta + P_m\left(-\frac{2x''}{\varepsilon}, 0, -1\right). \tag{15}$$

The restriction on P_m in condition (A) means that there is a positive constant C_3 such that

$$|P_m(X'', 0, -1)| \leqq C_3 |X''|^p,$$

for $|X''|$ sufficiently small. When k is even

$$\delta = t_0^{k-1+(\frac{1}{4})}, \qquad \frac{x_0}{\varepsilon} = t_0^{l-1/(2p)},$$

and taking $\sigma = +1$, we have

$$P_m(\operatorname{grad} F) \geqq (\tfrac{1}{2})\, a\, t_0^{k-1+\frac{1}{4}} - C_3\, 2^p\, t_0^{pl-\frac{1}{2}} > 0$$

in U for sufficiently small t_0, since $p\,l \geqq k$. When k is odd and $\sigma = 1$, we have

$$\delta = t_0^{k-1+1/(4p)}, \qquad \frac{x_0}{\varepsilon} = t_0^{l-2/p+1/(4p)},$$

so that

$$P_m(\operatorname{grad} F) \geqq (\tfrac{1}{2})\, a\, t_0^{k-1+1/(4p)} - C_3\, 2^p\, t_0^{pl-2+\frac{1}{4}} > 0$$

in U since $p\,l \geqq k+1$. In a similar way, when k is odd and $\sigma = -1$, we have $P_m(\operatorname{grad} F) < 0$ in U.

4. Proof of Theorem 2

The proof of Theorem 2 is identical with the proof of Theorem 1 with $p=2$ and $l=1$ except that in verifying condition (14) condition (B) is used in place of condition (A). In fact it follows from equation (15) that $P_m(\operatorname{grad} F) \neq 0$ in U provided t_0 is sufficiently small and the sign of σ agrees with the sign of $P_m(-2x''/\varepsilon, 0, -1)$ wherever it is not zero. When k is even, σ can be chosen as we please. When k is odd, σ has the sign of $[D_{n-1}\Phi(x)]_{x=0}$.

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